Sr.No. 4476

Exam Code: 211003
Subject Code: 4860

M.Sc. Mathematics - 3rd Sem.

(2119)

Paper : MATH-571

Functional Analysis-I

Time allowed : 3 hrs. **Max. Marks : 100**

Note: The question paper consists of **FOUR** sections. Each section contains **TWO** questions. The candidates are required to attempt **FIVE** questions selecting at least one question from each section. The fifth question may be attempted from any section. All questions carry equal marks.

SECTION - A

1. (a) State and prove Minkowski's inequality. (7)

   (b) Let $C = C[0,1]$ be the space of all continuous functions on $[0,1]$ and define $\|f\| = max|f(x)|$. Prove that C is a Banach space. (7)

   (c) If M is a closed linear subspace of a normed linear space N and T is the natural mapping of N onto N/M defined by $T(x) = x + M$, Show that T is continuous linear transformation for which $\|T\| \leq 1$. (6)

2. (a) State and prove Riesz- fischer theorem. (10)

   (b) Establish a version of the Holder inequality for $0 < p < 1$. (5)

   (c) Prove that the norms $\|T\| = sup\{\|T(x)\| : \|x\| = 1\}$ and $\|T\| = inf\{K : K \geq 0$ and $\|T(x)\| \leq K\|x\|$ for all $x\}$ are equivalent for a linear transformation T. (5)

SECTION - B

PTO

3. (a) If N is a normed linear space and x_0 is a non zero vector in N, then there exists a functional f_0 in N^* such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$. (6)

(b) State and prove Riesz theorem. (8)

(c) Prove that $l_1^* = l_\infty$. (6)

4. (a) State and prove Hahn- Banach theorem. (8)

(b) If B is a Banach space, prove that B is reflexive iff B^* is reflexive. (6)

(c) Prove that $\mathfrak{B}(N)$, the collection of all linear operators on N, is an algebra. (6)

Section - C

5. (a) State and prove Open mapping theorem. (7)

(b) Let T be an operator on a normed linear space N. If N is considered to be part of N^{**} by means of natural imbedding. Show that T^{**} is an extension of T. (6)

(c) If T is an operator on a normed linear space N, then its conjugate T^* defined by $(T^*(f))(x) = f(T(x))$ is an operator on N^* and the mapping $T \to T^*$ is an isometric isomorphism of $\mathfrak{B}(N)$ into $\mathfrak{B}(N^*)$, which reverses products and preserves the identity transformation. (7)

6. (a) Let B be a Banach space and M and N be closed linear subspaces of B such that $B = M \bigoplus N$. If $z = x + y$ is the unique representation of a vector in B as a sum of vectors in M and N, then the mapping P defined by $P(z) = x$ is a projection on B, whose range and null spaces are M and N. (7)

(b) State and prove Uniform boundedness principle. (6)

(c) A one-one continuous linear transformation of one Banach space into another is a homeomorphism. (7)

SECTION - D

7. (a) If M is a closed linear subspace of a Hilbert space H, then $H = M \bigoplus M^{\perp}$. (8)

(b) State and prove Parallelogram law. (3)

(c) Prove that l_2^n is a Hilbert space. (5)

(d) If M is a linear subspace of a Hilbert space. Show that M is closed iff $M = M^{\perp\perp}$. (4)

8. (a) State and prove Bessel's inequality. (6)

(b) Let H be a Hilbert space and f be an arbitrary functional in H^*. Then there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H. (6)

(c) If M is a closed linear subspace of a Hilbert space H and x be a vector not in M and let d be the distance from x to M. Then there exists a unique vector y_0 in M such that $\|x - y_0\| = d$. (4)

(d) Show that the Parallelogram law is not true in l_1^n, $n > 1$. (4)

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